TECHNICAL RESEARCH REPORT

Computation for Nonlinear Balancing

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T.R. 98-40

INSTITUTE FOR SYSTEMS RESEARCH

Report Documentation Page

Form Approved
OMB No. 0704-0188

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1. REPORT DATE **1998**	2. REPORT TYPE	3. DATES COVERED **-**

4. TITLE AND SUBTITLE **Computation for Nonlinear Balancing**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Army Research Office,PO Box 12211,Research Triangle Park,NC,27709**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES
The original document contains color images.

14. ABSTRACT
see report

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES **3**	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**			

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

Computation for Nonlinear Balancing

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Abstract

We illustrate a computational approach to practical nonlinear balancing via the forced damped pendulum example.

1 Introduction

The theory of balancing, introduced by Moore [1], was extended by Scherpen [2] to a class of stable finite–dimensional systems in the nonlinear setting

$$\dot{x} = f(x) + g(x)\,u \qquad (1)$$
$$y = h(x) \qquad (2)$$

where $x = (x_1, \ldots, x_n)$ are local coordinates for a smooth state manifold, f, g, and h are of class C^∞, $f(0) = 0$, and $h(0) = 0$.

The existence of efficient computational methods has been crucial in making balancing an effective and important tool for model reduction of linear systems. The balancing change of coordinates can be computed using established matrix equation and decomposition algorithms. In contrast, Scherpen's procedure for nonlinear balancing presents computational difficulties. In this paper we address these difficulties and illustrate computational methods by means of an example.

2 Balancing for the Forced Damped Pendulum

We model a damped pendulum with torque input and measured angular position by

$$\dot{x}_1 = x_2 ; \quad \dot{x}_2 = -a\sin(x_1) - b\,x_2 + c\,u \quad (3)$$
$$y = x_1 \qquad (4)$$

where we choose $a = 3$, $b = 1$, and $c = 0.1$.

The main objects involved in Scherpen's theory are the controllability and observability energy functions, $L_c : \mathbb{R}^n \to \mathbb{R}$ and $L_o : \mathbb{R}^n \to \mathbb{R}$, respectively, defined by

$$L_c(x_0) = \min_{\substack{u \in \mathcal{L}_2(-\infty, 0) \\ x(-\infty) = 0 \\ x(0) = x_0}} \frac{1}{2} \int_{-\infty}^{0} \|u(t)\|^2 \, dt \qquad (5)$$

and

$$L_o(x_0) = \frac{1}{2} \int_{0}^{\infty} \|y(t)\|^2 \, dt\,, \qquad (6)$$
$$x(0) = x_0\,, \quad u(t) \equiv 0\,, \quad 0 \le t < \infty.$$

Computation of L_c via (5) requires solution of an optimal control problem at each point on a state space grid. One alternative is to numerically solve the Hamilton–Jacobi type partial differential equation (PDE) given by [2] eqn. (12). Both of these methods are computationally intensive (if a numerical solution can be generated at all) and require apriori knowledge of f and g. We offer an empirical approach which could be used in the absence of accurate models.

In the linear case, $L_c(x) = \frac{1}{2}\,x^T W_c^{-1}\,x$, is quadratic with W_c denoting the controllability Gramian matrix. Injection of Gaussian white noise at the input terminals produces a Gaussian asymptotic probability density function (PDF) for the state

$$p_\infty(x) = [(2\pi)^n \det(R_\infty)]^{-1/2} \exp(-L_c(x)) \qquad (7)$$

where $R_\infty = W_c$ denotes the asymptotic state covariance. Therefore, in the nonlinear setting, we estimate L_c via

$$\widehat{L}_c(x) = -\log(p_\infty(x)) + \text{constant.} \qquad (8)$$

This suggests an empirical Monte–Carlo approach in which the system response to Gaussian white noise is recorded for a large number of sample paths, and the data is histogrammed on an appropriate grid to compute a PDF and estimate for L_c. The empirical approach is also used for determination of L_o via numerical integration of (6) where the input signal is null and

the initial condition x_0 takes each value in the state space grid. Figures 1–3 show results for the pendulum system (3–4).
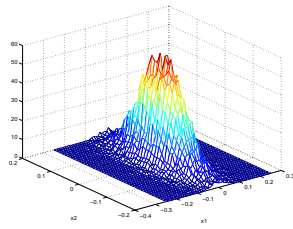

Figure 1: Probability density function for asymptotic state of pendulum system.
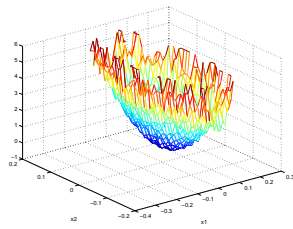

Figure 2: Estimated controllability function for pendulum system.
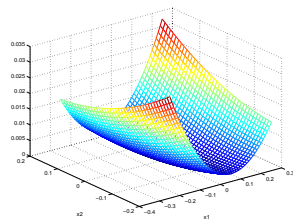

Figure 3: Observability function for pendulum system.

The existence of the balancing transformation is proved using the Morse–Palais lemma (see [3, 4]) which asserts that there exists a change of coordinates under which a function is locally quadratic in a neighborhood $\mathcal{U}$ of a nondegenerate critical point at 0, i.e., $f(x) = < A\,\phi(x),\, \phi(x) >$, with ϕ a diffeomorphism defined on $\mathcal{U}$. Computation of ϕ requires the decomposition $f(x) = < H(x)\,x,\, x >$ (see [3]). It is shown in [4] that $\phi(x) = C(x)\,x$ where $C(x)^2 = B(x) = H(0)^{-1}\,H(x)$. Such a C exists on a neighborhood of 0 since $B(0)$ is the identity and a square root function is defined in a neighborhood of the identity operator $\mathbb{1}$ by a convergent power series. The algorithm we use is

$$S_0 = 0\,,\ S_{k+1} = \frac{1}{2}\left[(\mathbb{1} - B) + S_k^2\right],\ k = 0, 1, \ldots \quad (9)$$

with $C = \mathbb{1} - S_\infty$. It is valid for x such that $\|\mathbb{1} - B(x)\| < 1$ thus providing an estimate of the neighborhood $\mathcal{U}$.

The relative input–output influence of each state in the balanced realization is measured by the singular value functions (SVF) (see [2]), which are the point–dependent analogs of the Hankel singular values for a linear system. Figure 4 shows the resulting SVFs for the pendulum system (3–4). We see that in the balanced realization, one state has roughly twice the input–output influence as the other. The values of the singular value functions at the origin are 0.0376 and 0.0193, respectively. As expected, these are in close agreement with the Hankel singular values of the linearized system, which are 0.0384 and 0.0217, respectively.
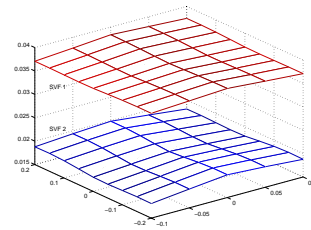

Figure 4: Singular value functions for pendulum system in a small neighborhood of the origin (denoted SVF 1 and SVF 2).

3 Conclusion

We have illustrated a computational approach for practical implementation of Scherpen's nonlinear balancing procedure via application to the forced damped pendulum system. A Monte–Carlo approach provides an estimate of the controllability energy function. The observability energy function is computed via numerical integration and is also suitable as an empirical method. Numerical implementation of the Morse-Palais lemma is achieved using a successive approximation algorithm for an operator square root which appears in Palais' proof. The corresponding convergence criterion gives an estimate of the neighborhood in which the given function is locally quadratic. The balanced realization for the pendulum system results in one state having roughly double the input–output influence as the other state.

References

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